INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2009; **77**:1863–1869
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.2587

LETTER TO THE EDITOR

On the essence and the evaluation of the shape functions for the smoothed finite element method (SFEM)

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SUMMARY

This paper is written in response to the recently published paper (*Int. J. Numer. Meth. Engng* 2008; **76**:1285–1295) at IJNME entitled 'On the smoothed finite element method' (SFEM) by Zhang HH, Liu SJ, Li LX.

In this paper we

(1) repeat briefly the important essence of the original SFEM presented in (*Comp. Mech.* 2007; **39**: 859–877; *Int. J. Numer. Meth. Engng* 2007; **71**:902–930; *Int. J. Numer. Meth. Engng* 2008; **74**:175–208; *Finite Elem. Anal. Des.* 2007; **43**:847–860; *J. Sound Vib.* 2007; **301**:803–820), and
(2) examine further issues in the evaluation of the shape functions used in the SFEM.

It will be shown that the 'SFEM' presented in paper (*Int. J. Numer. Meth. Engng* 2008; **76**:1285–1295) is *not at all* our original SFEM presented in (*Comp. Mech.* 2007; **39**:859–877; *Int. J. Numer. Meth. Engng* 2007; **71**:902–930; *Int. J. Numer. Meth. Engng* 2008; **74**:175–208; *Finite Elem. Anal. Des.* 2007; **43**:847–860; *J. Sound Vib.* 2007; **301**:803–820). Therefore, all these 'Theorems', 'Corollaries' and 'Remarks' presented in paper (*Int. J. Numer. Meth. Engng* 2008; **76**:1285–1295) have nothing to do with our original SFEM. The properties of the original SFEM stand as they were presented in our original papers (*Comp. Mech.* 2007; **39**:859–877; *Int. J. Numer. Meth. Engng* 2007; **71**:902–930; *Int. J. Numer. Meth. Engng* 2008; **74**:175–208; *Finite Elem. Anal. Des.* 2007; **43**:847–860; *J. Sound Vib.* 2007; **301**:803–820). Finally, we brief on our advancements made far beyond our original SFEM and our visions on future numerical methods. Copyright © 2009 John Wiley & Sons, Ltd.

Received 18 December 2008; Revised 22 December 2008; Accepted 30 January 2009

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KEY WORDS: numerical methods; meshfree methods; finite element method (FEM); smoothed finite element method (SFEM); strain smoothing operator; linear point interpolation

1. BACKGROUND

In 2007, Liu *et al.* [1] proposed a smoothed finite element method (SFEM) to improve the performance of the standard finite element method (FEM) using four-node quadrilateral elements. The SFEM combines the existing FEM technology and the strain smoothing technique used in meshfree methods [2]. The essence of the SFEM is that *only shape function values at points on the boundaries of the smoothing cells* are needed and only the compatibility of the nodal shape functions on the boundaries of these smoothing cells is required in the formulation. This gives tremendous freedom to compute these shape function values for the SFEM, and they can be easily obtained using the simple linear point interpolation method (PIM) without mapping [1].

Numerical studies in [1] have demonstrated that SFEM shows some superiorities over the standard FEM using four-node isoparametric elements. For example, (1) the SFEM model is softer than the FEM model using the same mesh; (2) SFEM can give more accurate solutions than that of FEM in both displacement and energy norms; (3) domain discretization of SFEM is much more flexible than the FEM and heavily distorted, tile and polygonal elements can be used; and (4) construction of shape functions in our SFEM can be much easier than in the FEM by using simple linear point interpolation. These good features can be gained without increasing much the efforts both in modeling and computation, and the SFEM is also easy to implement into the existing FEM code with little alterations/modifications.

The related theory of SFEM was also set up and properly proven in detail [3, 4]. The SFEM has also been developed for general n-sided polygonal elements (nSFEM) [5], dynamic analyses [6] and further extended for plate and shell analyses [7–11], and piezoelectric structures [12].

2. THE DIFFERENCE BETWEEN THE ORIGINAL SFEM AND THE 'SFEM' PRESENTED IN [13]

In *theory*, paper [13] omitted the crucial theoretical essence of the original SFEM: the smoothed strains are obtained using ONLY continuous shape function values and ONLY at discrete points on the boundaries of the smoothing cells. The exact expression of the shape functions over the entire element (or each smoothing cell) is *immaterial* and we do *not need to know* (it can be, in general, very complicated and cannot be given in explicit forms). On the boundary of the smoothing cells, however, the variation of the shape functions is up to us to determine (linearly or otherwise), as long as the continuity of the shape functions on the boundaries of the smoothing cells is ensured. This is also the reason why we do not need mapping as we do in the standard FEM.

In *implementation*, the vital difference is the evaluation of the shape functions. In ANY of these original papers [1, 3–6], the shape function values at points (vertices and mid-points of cell edges of the smoothing cells) are obtained using simple linear point interpolations following the essence of the SFEM theory. Even the detailed interpolated values were specifically provided in a very clear manner either in tabular form (e.g. Table 1 in [1]) or in picture form (e.g. Figure 1 in [6]) or in combined tabular and picture forms (e.g. Figure 4 in [5]). It is not possible to be mistaken.

Paper [13] omitted all these essential and important numerical techniques, and the shape functions defined in Equation (22) in [1] were used in a different way with largely the same conventional thinking in the standard FEM to formulate their 'SFEM'.

It is true that the shape functions (Equation (22) in [1]) were presented in the original SFEM paper [1], and it was stated that it *can* be used for computing the shape function values for an *interior* point (second line before Equation (21) in [1]). It was also stated that Equation (22) in [1] is not used because of the *inconvenience* (last line in page 863 in [1]). All these statements are correct and not possible to be mistaken. Paper [13] also omitted all these statements given in the original paper.

We re-claim that Equation (22) in [1] can be used, but not in the way stated in paper [13]. As long as the essence of the original SFEM is respected, there are plenty of ways to make good use of it. The essential rule is that it can only be used to interpolate the values of shape function for an interior point (that is usually the vertices and mid-points of *interior* cell edges on the smoothing cells) strictly within the element, as clearly stated in paper [1]. We cannot (in general) use it for any of the points on the cell edges that are right on the boundary of the element. For any points on the boundaries of the smoothing cells that are also on the boundary of the element, the values like those given in the tables or pictures presented in the original papers should be used.[‡] This is to ensure the compatibility of the shape functions on the boundaries of all these smoothing cells.

Of course, when Equation (22) in [1] is used, there can be the singularity issue with the moment matrix that was well known a long time ago, and was discussed in detail in the textbook [14] with a number of possible remedies. In this *particular* case (only with a rank deficiency of 1) of using Equation (22) in [1], we only need to use the coordinate transformation technique detailed in [14] to overcome the singularity issue, and this technique always works, as long as four nodes of the element are not in-line (form geometrically a quadrilateral element by definition). However, in this case, the procedure for computing the shape functions for the SFEM becomes more complicated, inconvenient to use, and hence was not used in any of these papers on the original SFEM. For the interest of many and for other good uses of Equation (22) in [1], we detailed a procedure in Section 3, and show that the technique of using Equation (22) in [1] together with the coordinate transformation works very well as long as we follow the essence of the original SFEM and perform it in a proper manner.

Because paper [13] omitted all these essential theories, important numerical techniques and the statements presented in our papers on the original SFEM, all these 'Theorems', 'Corollaries' and 'Remarks' presented in paper [13] do not apply to our original SFEM. The properties of the original SFEM stand as they were presented in our original papers [1, 3–6].

3. THE PROPER USE OF EQUATION (22) IN [1]

Note again that for all the points on the cell edges that are on the element boundary, the shape function values should be obtained using the tables and pictures presented in the papers on the original SFEM [1, 3–6]. Equation (22) in [1] is used only for points that are strictly inside the element.

[‡] Plenty of other ways can also be used as long as the essence of the SFEM is respected, but it is not the scope of this paper.

DOI: 10.1002/nme

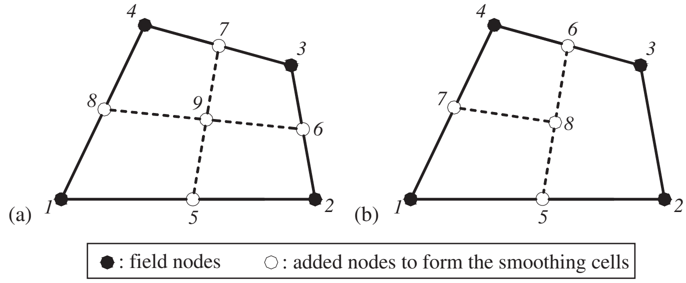

Figure 1. Division of element into four and three smoothing cells (SC) in the SFEM. (a) Four smoothing cells and (b) Three smoothing cells.

We now note that in the original SFEM for four-node quadrilaterals (with, for example, SC = 3 or 4), we need only to determine the shape function values for an interior point (say, for example, point 9 in Figure 1(a) or point 8 in Figure 1(b)) that is a vertex of the smoothing cells. The value of the shape function at any vertex of smoothing cells $\mathbf{x}_e = [x_e \ \ y_e]^{\mathrm{T}}$ located (strictly) *inside* the quadrilateral element can be obtained by using Equation (22) in [1], which is re-written as

$$N(\mathbf{x}_e) = \underbrace{[1 \ \ x_e \ \ y_e \ \ x_e y_e]}_{\mathbf{p}^{\mathrm{T}}} \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1}}_{\mathbf{a}} \tag{1}$$

where $\mathbf{x}_i = [x_i \ \ y_i]^{\mathrm{T}}$ $(i = 1, \ 2, \ 3, \ 4)$ are the coordinates of the four nodes associated with this element. Equation (1) assumes that the inversion of the moment matrix $\mathbf{a}$ exists.

It is clear that the value of shape functions at endpoints $\mathbf{x}_e$ satisfies: (i) partition of unity: $\sum_{i=1}^{n} \bar{N}_i(\mathbf{x}_e) = 1$; (ii) linear consistency: $\sum_{i=1}^{n} \bar{N}_i(\mathbf{x}_e)\mathbf{x}_i = \mathbf{x}_e$, as along as $\mathbf{a}$ is invertible (see proof in [14]). It has been well known for long that the inversion of the matrix $\mathbf{a}$ in Equation (1) may not exist as discussed in the textbook [14]. In these cases, we apply the coordinate transformation technique given in [14] (see, Figure 2):

$$\begin{aligned} \xi &= (x - x_e)\cos\varphi + (y - y_e)\sin\varphi \\ \eta &= -(x - x_e)\sin\varphi + (y - y_e)\cos\varphi \end{aligned} \tag{2}$$

where (x_e, y_e) is the origin of the local coordinate system (ξ, η), which is defined in a global coordinate system $(x, \ y)$, and $\varphi \in [\pi/6, \pi/3]$ is the rotation angle for the local coordinate system (ξ, η) with respect to the global coordinate system. Applying the above transformation, the values of shape functions at $\mathbf{x}_e = [x_e \ \ y_e]^{\mathrm{T}}$ can always be obtained using a $\varphi \in [\pi/6, \pi/3]$, as long as these four nodes of the element are not in-line.

DOI: 10.1002/nme

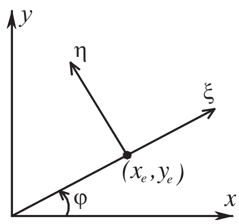

Figure 2. A local coordinate system (ξ, η) defined in a global coordinate system (x, y). The origin of (ξ, η) is located at (x_e, y_e).

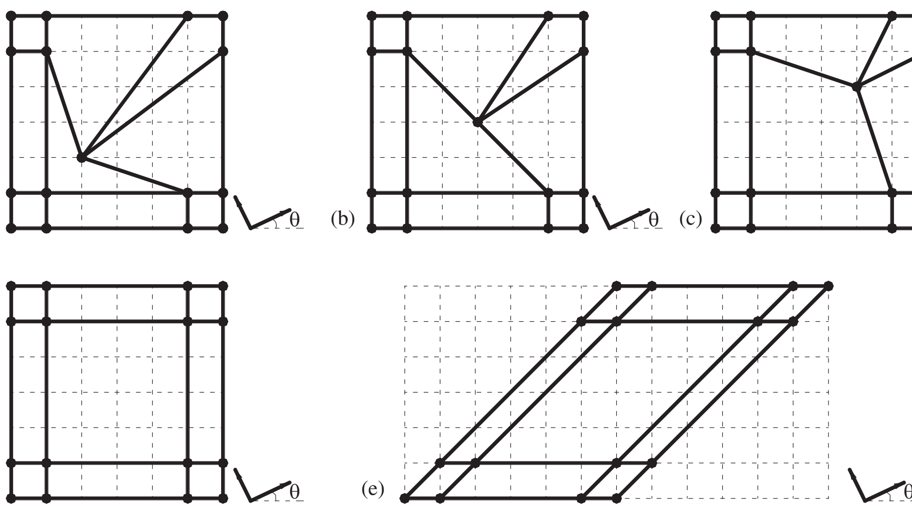

Figure 3. Meshes used for the patch test: (a) a mesh with a concave quadrilateral element; (b) a mesh with a quadrilateral element using three collinear points; (c) a mesh with general convex quadrilateral elements; (d) a mesh with rectangular elements; and (e) a mesh with parallelogram elements.

Applying Equation (2) to the four field nodes of the quadrilateral element and the point $\mathbf{x}_e = [x_e \;\; y_e]^{\mathrm{T}}$, Equation (1) now becomes

$$N(\mathbf{x}_e) = [1 \;\; 0 \;\; 0 \;\; 0] \begin{bmatrix} 1 & \xi_1 & \eta_1 & \xi_1\eta_1 \\ 1 & \xi_2 & \eta_2 & \xi_2\eta_2 \\ 1 & \xi_3 & \eta_3 & \xi_3\eta_3 \\ 1 & \xi_4 & \eta_4 & \xi_4\eta_4 \end{bmatrix}^{-1} \tag{3}$$

The standard patch tests, using our SFEM, are conducted for five patches shown in Figure 3 that includes all the cases used in paper [13] to show the properties of the 'SFEM'. In our tests using the original SFEM, we created a total of 25 test cases by rotating each of these five patches shown in Figure 3 with $\theta = 0$, $\pi/6$, $\pi/4$, $\pi/3$ and $\pi/2$. Three or four smoothing cells shown in Figure 1 are used for each quadrilateral element used in the each of these patches. For the points

DOI: 10.1002/nme

on the boundary of the element, the shape function values are obtained using those given in Table 1 in [1]. For the vertices of smoothing cells located *inside* the quadrilateral elements, the shape function values are obtained by using the interpolation Equation (1) is used. During the computation, the determinate of the matrix **a** in Equation (1) is always checked. If the absolute value of the determinant of the matrix **a** is larger than a predefined critical value, Equation (1) is used, or else Equations (2) and (3) are used. The results show that the SFEM always passes the standard patch test for all these 25 cases.

4. OTHER ISSUES

The essence of the SFEM reveals that we only need to ensure the continuity of the shape functions on the boundaries of the smoothing cells. This has led to the further development of the node-based smoothed FEM or NS-FEM [15] that can often produce the important upper bound solutions for force-driven 2D and 3D solid mechanics problems. The NS-FEM was found, however, 'overly soft', which can lead to spurious non-zero energy modes for dynamic problems. This has led to the development of edge-based FEM or ES-FEM [16] that is often found 'ultra-accurate' (one order more accurate than the FEM model using the same three-node triangular mesh) for 2D elasticity problems and applicable to both static and dynamic analyses. For the 3D problems, ES-FEM is extended into face-based FEM or FS-FEM [17], following the same essential theories and techniques of the original SFEM.

The gradient smoothing technique is further generalized for discontinuous functions [18]. The generalized smoothing technique forms the theoretical foundation for NS-PIM [19], NS-RPIM [18–21], ES-PIM and ES-RPIM [22] where incompatible PIM shape functions were used to establish conforming models. The variation of the shape functions along the boundary of the smoothing cells is in general of arbitrary order. Our pursuit in this direction is still ongoing, and we are confident that much more effective numerical models will be established. We hope that our long-term, indomitable and persistent efforts in using shape functions regardless of their incompatibility to conveniently create efficient numerical models will contribute significantly in the general area of numerical methods. Our vision is that as long as a set of independent shape functions in a proper G space [18] can be created in either FEM or meshfree settings, a stable and convergent numerical model can always be established via a proper weakened weak formulation. The compatibility of the shape functions is no longer an issue, which allows much more flexible ways to create shape functions for much more efficient numerical methods.

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